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On the rational terms of the one-loop amplitudes

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Abstract: The various sources of Rational Terms contributing to the one-loop amplitudes are critically discussed. We show that the terms originating from the generic $(n-4)-$dimensional structure of the numerator of the one-loop amplitude can be derived by using appropriate Feynman rules within a tree-like computation. For the terms that originate from the reduction of the $4-$dimensional part of the numerator, we present two different strategies and explicit algorithms to compute them.

Keywords: Standard Model, NLO Computations, QCD, Hadronic Colliders.

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Contents

1. Introduction

In the last few years, a big effort has been devoted by several authors to the problem of computing one-loop amplitudes efficiently [1]. Besides Standard techniques, where tensor reduction/computation is performed, numerically or analytically, new developments emerged, originally inspired by unitarity arguments (the so called unitary and generalized unitarity methods) [2]–[5], in which the tensor reduction/computation is substituted by the problem of determining the *coefficients* of the contributing scalar one-loop functions. This possibility relies on the fact that the basis of one-loop integrals is known in terms of Boxes, Triangles, Bubbles and (in massive theories) Tadpoles, in such a way that, schematically, one can write a Master Equation for any one-loop amplitude $\mathcal{M}$ such as:

$$\mathcal{M} = \sum_i d_i \text{ Box}_i + \sum_i c_i \text{ Triangle}_i + \sum_i b_i \text{ Bubble}_i + \sum_i a_i \text{ Tadpole}_i + \mathrm{R}\,, \tag{1.1}$$

where d_i, c_i, b_i and a_i are the coefficients to be determined.

Nevertheless, in practice, only the part of the amplitude proportional to the one-loop scalar functions can be obtained straightforwardly in the unitary cut method. The remaining Rational Terms, RTs, (R in the above equation) should be reconstructed by other means, either by direct computation [6, 7] or by boostrapping methods [8]. In [9] the RTs are obtained by explicitly computing the amplitude at different integer value of the space-time dimensions. In [10] cut-constructible and rational parts are calculated at the same time by using d-dimensional unitarity cuts.

On the other hand, in another recently proposed method, `OPP` [11], a class of terms contributing to R can be naturally derived in the same framework used to determine all the other coefficients.

In this paper, we critically analyze the various sources of RTs appearing in one-loop amplitudes, by classifying them in two categories: $\mathrm{R} = \mathrm{R}_1 + \mathrm{R}_2$, also presenting a few computational methods. In the next section, we investigate the origin of R_2 and develop a practical computational strategy. In section 3, after a brief recall of the `OPP` method, we present a way to compute R_1, strictly connected to the `OPP` framework. In section 4, we describe yet another method that relies on cuts in n-dimensions. This second method is proven more suitable for numerical applications within the `OPP` algorithm and may also be applied in a more general framework. Finally, in the last section, we outline our conclusions.

2. The origin of R_2

Our starting point is the general expression for the *integrand* of a generic m-point one-loop (sub-)amplitude [11]

$$\bar{A}(\bar{q}) = \frac{\bar{N}(\bar{q})}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}\,, \quad \bar{D}_i = (\bar{q} + p_i)^2 - m_i^2\,, \quad p_0 \neq 0\,. \tag{2.1}$$

In the previous equation, dimensional regularization is assumed, so that we use a bar to denote objects living in $n = \ 4 + \epsilon$ dimensions. Furthermore $\bar{q}^2 = q^2 + \tilde{q}^2$, where $\tilde{q}^2$ is ϵ-dimensional and $(\tilde{q} \cdot q) = 0$. The numerator function $\bar{N}(\bar{q})$ can be also split into a 4-dimensional plus a ϵ-dimensional part

$$\bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q}^2, q, \epsilon)\,. \tag{2.2}$$

$N(q)$ is 4-dimensional (and will be discussed in the next section) while $\tilde{N}(\tilde{q}^2, q, \epsilon)$ gives rise to the RTs of kind R_2, defined as

$$\mathrm{R}_2 \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q} \frac{\tilde{N}(\tilde{q}^2, q, \epsilon)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q}\,\mathcal{R}_2\,. \tag{2.3}$$

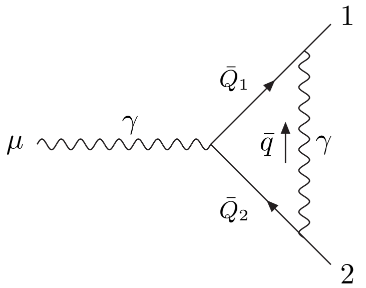
To investigate the explicit form of $\tilde{N}(\tilde{q}^2, q, \epsilon)$ it is important to understand better the separation in eq. (2.2). From a given *integrand* $\bar{A}(\bar{q})$ this is obtained by splitting, in the numerator function, the n-dimensional integration momentum $\bar{q}$, the n-dimensional γ matrices $\bar{\gamma}_{\bar{\mu}}$ and the n-dimensional metric tensor $\bar{g}^{\bar{\mu}\bar{\nu}}$ into a 4-dimensional component plus remaining pieces:

$$\begin{aligned} \bar{q} &= q + \tilde{q}\,, \\ \bar{\gamma}_{\bar{\mu}} &= \gamma_\mu + \tilde{\gamma}_{\tilde{\mu}}\,, \\ \bar{g}^{\bar{\mu}\bar{\nu}} &= g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}}\,. \end{aligned} \tag{2.4}$$

Notice that, when a n-dimensional index is contracted with a 4-dimensional (observable) vector v_μ, the 4-dimensional part is automatically selected. For example

$$\bar{q} \cdot v = q \cdot v \quad \text{and} \quad \bar{\not{v}} = \not{v}\,. \tag{2.5}$$

A practical way to compute R_2 is determining, once for all, tree-level like Feynman Rules for the theory at hand by calculating, with the help eq. (2.4), the R_2 part coming from



$$\bar{Q}_1 = \bar{q} + p_1 = Q_1 + \tilde{q}$$
$$\bar{Q}_2 = \bar{q} + p_2 = Q_2 + \tilde{q}$$

$$\bar{D}_0 = \bar{q}^2$$
$$\bar{D}_1 = (\bar{q} + p_1)^2$$
$$\bar{D}_2 = (\bar{q} + p_2)^2$$

Figure 1: QED $\gamma e^+ e^-$ diagram in n dimensions.

one-particle irreducible amplitudes up to four external legs. The fact that four external legs are enough is guaranteed by the ultraviolet nature of the RTs, proven in [6].

As an illustrative example, we derive the complete set of the Feynman Rules needed in QED. Along such a line R_2 can be straightforwardly computed in any theory. We start from the one-loop $\gamma e^+ e^-$ amplitude in figure 1. The numerator can be written as follows

$$\begin{aligned} \bar{N}(\bar{q}) &\equiv e^3 \left\{ \bar{\gamma}_{\bar{\beta}} \, (\bar{\not{Q}}_1 + m_e) \, \gamma_\mu \, (\bar{\not{Q}}_2 + m_e) \, \bar{\gamma}^{\bar{\beta}} \right\} \\ &= e^3 \left\{ \gamma_\beta (\not{Q}_1 + m_e) \gamma_\mu (\not{Q}_2 + m_e) \gamma^\beta \right. \\ &\quad \left. - \epsilon \, (\not{Q}_1 - m_e) \gamma_\mu (\not{Q}_2 - m_e) + \epsilon \tilde{q}^2 \, \gamma_\mu - \tilde{q}^2 \, \gamma_\beta \gamma_\mu \gamma^\beta \right\} , \end{aligned} \tag{2.6}$$

where all ϵ-dimensional γ-algebra has been explicitly worked out[1] in order to get the desired splitting. The first term in the r.h.s. of eq. (2.6) is $N(q)$, while the sum of the remaining three define $\tilde{N}(\tilde{q}^2, q, \epsilon)$ for the case at hand. By inserting $\tilde{N}(\tilde{q}^2, q, \epsilon)$ in eq. (2.3) and using the fact that

$$\begin{aligned} \int d^n \bar{q} \frac{\tilde{q}^2}{\bar{D}_0 \bar{D}_1 \bar{D}_2} &= -\frac{i\pi^2}{2} + \mathcal{O}(\epsilon) , \\ \int d^n \bar{q} \frac{q_\mu q_\nu}{\bar{D}_0 \bar{D}_1 \bar{D}_2} &= -\frac{i\pi^2}{2\epsilon} g_{\mu\nu} + \mathcal{O}(1) , \end{aligned} \tag{2.7}$$

gives

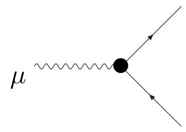
$$\mathrm{R}_2 = -\frac{ie^3}{8\pi^2} \gamma_\mu + \mathcal{O}(\epsilon) , \tag{2.8}$$

that can be used to define the effective vertex of figure 2.

With analogous techniques, taking also into account the integrals given in eq. (4.2), one gets all the remaining QED effective vertices given in figure 3.

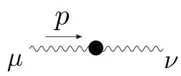
To summarize, the problem of computing R_2 is reduced to a tree level calculation and we consider it fully solved. The R_1 part is, instead, deeply connected to the structure of the one-loop amplitude, as we shall see in the next section. It is worthwhile to mention that only the full $\mathrm{R} = \mathrm{R}_1 + \mathrm{R}_2$ constitutes a physical gauge-invariant quantity in dimensional regularization.

[1] ϵ-dimensional γ matrices freely anti-commute with four-dimensional ones: $\{\gamma_\mu, \tilde{\gamma}_\nu\} = 0$.

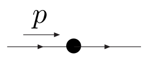


$$= -\frac{ie^3}{8\pi^2}\gamma_\mu$$

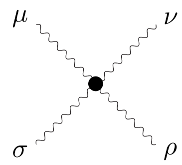
Figure 2: QED $\gamma e^+ e^-$ effective vertex contributing to R_2.



$$= -\frac{ie^2}{8\pi^2}\, g_{\mu\nu}\,(2m_e^2 - p^2/3)$$



$$= \frac{ie^2}{16\pi^2}(-\not{p} + 2m_e)$$



$$= \frac{ie^4}{12\pi^2}\,(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho})$$

Figure 3: QED $\gamma\gamma$, ee and $\gamma\gamma\gamma\gamma$ effective vertices contributing to R_2.

3. The OPP method and the origin of $\mathbf{R_1}$

The OPP reduction algorithm provides a useful framework to understand the origin of the RTs of kind R_1. The starting point of the method is an expansion of $N(q)$ in terms of 4-dimensional denominators $D_i = (q + p_i)^2 - m_i^2$

$$\begin{aligned}
N(q) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i \neq i_0}^{m-1} D_i \\
& + \tilde{P}(q) \prod_{i}^{m-1} D_i \,.
\end{aligned} \tag{3.1}$$

Inserted back in eq. (2.1), this expression simply states the multi-pole nature of any m-point one-loop amplitude. The fact that only terms up to 4 poles appear is due to the fact that m-point scalar loop functions with $m > 4$ are always expressible in terms of boxes up to

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contributions $\mathcal{O}(\epsilon)$. The last term with no poles, $\tilde{P}(q)$, has been inserted for generality, but is zero in practical calculations where m-point amplitudes behave such as $N(\lambda q) \to \lambda^m$ when $\lambda \to \infty$. The coefficients of the poles can be further split in two pieces. A piece that still depend on q (the terms $\tilde{d}, \tilde{c}, \tilde{b}, \tilde{a}$), that vanishes upon integration due to Lorentz invariance, and a piece that do not depend on q (the terms d, c, b, a). Such a separation is always possible, as shown in [11], and, with this choice, the latter set of coefficients is therefore immediately interpretable as the ensemble of the coefficients of all possible 4, 3, 2, 1-point one-loop functions contributing to the amplitude.

Once eq. (3.1) is established, the task of computing the one-loop amplitude is then reduced, in the `OPP` method, to the algebraical problem of fitting the coefficients d, c, b, a by evaluating the function $N(q)$ a sufficient number of times, at different values of q, and then inverting the system. Notice that this can be performed *at the amplitude level* and that one does not need to repeat the work for all Feynman diagrams, provided their sum is known.

The `OPP` expansion is written in terms of 4-dimensional denominators. On the other hand, n-dimensional denominators $\bar{D}_i$ appear in eq. (2.1), that differ by an amount $\tilde{q}^2$ from their 4-dimensional counterparts

$$\bar{D}_i = D_i + \tilde{q}^2 \tag{3.2}$$

The result of this is a mismatch in the cancellation of the n-dimensional denominators of eq. (2.1) with the 4-dimensional ones of eq. (3.1) (the `OPP` expansion), that originates a Rational Part. In fact, by inserting eq. (3.2) into eq. (3.1), one can rewrite it in terms of n-dimensional denominators (therefore restoring the exact cancellation), but at the price of adding an extra piece $f(\tilde{q}^2, q)$. The RTs of kind R_1 are defined as

$$\mathrm{R}_1 \equiv \frac{1}{(2\pi)^4} \int d^n\, \bar{q} \frac{f(\tilde{q}^2, q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \,. \tag{3.3}$$

The explicit form of the function $f(\tilde{q}^2, q)$ can be easily and explicitly obtained, in the framework of the `OPP` method, by rewriting any denominator appearing in eq. (2.1) as follows

$$\frac{1}{\bar{D}_i} = \frac{\bar{Z}_i}{D_i} \,, \quad \text{with} \quad \bar{Z}_i \equiv \left(1 - \frac{\tilde{q}^2}{\bar{D}_i}\right) . \tag{3.4}$$

This results in

$$\bar{A}(\bar{q}) = \frac{N(q)}{D_0 D_1 \cdots D_{m-1}} \bar{Z}_0 \bar{Z}_1 \cdots \bar{Z}_{m-1} + \mathcal{R}_2 \,, \tag{3.5}$$

where $\mathcal{R}_2$ is the integrand function introduced in eq. (2.3). Then, by inserting eq. (3.1) in eq. (3.5), one obtains

$$\begin{aligned}\bar{A}(\bar{q}) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0i_1i_2i_3) + \tilde{d}(q;i_0i_1i_2i_3)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}\bar{D}_{i_3}} \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} \bar{Z}_i \\ & + \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0i_1i_2) + \tilde{c}(q;i_0i_1i_2)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}} \prod_{i\neq i_0,i_1,i_2}^{m-1} \bar{Z}_i \\ & + \sum_{i_0<i_1}^{m-1} \frac{b(i_0i_1) + \tilde{b}(q;i_0i_1)}{\bar{D}_{i_0}\bar{D}_{i_1}} \prod_{i\neq i_0,i_1}^{m-1} \bar{Z}_i \\ & + \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q;i_0)}{\bar{D}_{i_0}} \prod_{i\neq i_0}^{m-1} \bar{Z}_i \\ & + \tilde{P}(q) \prod_{i}^{m-1} \bar{Z}_i + \mathcal{R}_2 \,. \end{aligned} \tag{3.6}$$

R_1 is then produced, after integrating over $d^n\bar{q}$, by the $\tilde{q}^2$ dependence coming from the various $\bar{Z}_i$ in eq. (3.6). This strategy have been adopted in [12], where also all needed integrals have been carefully classified and computed.

Although rather transparent, the above derivation of R_1 has two drawbacks. First of all, it requires the knowledge of the spurious terms.[2] Secondly, it is not suitable when combining diagrams together because, when taking common denominators, additional terms containing $\tilde{q}^2$ appear in the numerator, that may give rise to new rational parts. The book-keeping of such new structures is equivalent to the treatment of each diagram separately, jeopardizing the ability of the 4-dimensional `OPP` technique of dealing directly with the amplitude. For these reasons we present, in the next section, a different way of attacking this problem that does not relies on spurious terms and that also allows one to combine diagrams *before* fitting the coefficients d, c, b, a. This second method is better suited for a numerical implementation, and it has been already successfully implemented in a Fortran code [13].

4. The n-dimensional cuttings to compute R_1

The Rational Terms R_1 can be computed by looking at the implicit mass dependence (namely reconstructing powers of $\tilde{q}^2$) in the coefficients d, c, b of the one-loop functions, once $\tilde{q}^2$ is reintroduced through the mass shift

$$m_i^2 \to m_i^2 - \tilde{q}^2 \,. \tag{4.1}$$

This procedure is formally equivalent, in the generalized unitarity framework, to the applications of n-dimensional cuts, and is obtained, in the `OPP` language, by simply performing the `OPP` expansion of eq. (3.1) directly in terms of the n-dimensional denominators of eq. (3.2). By doing so, all coefficients of the `OPP` expansion start depending on $\tilde{q}^2$. The

[2] After multiplication with the $\bar{Z}_i$, they give non vanishing contributions.

spurious terms keep being spurious, because they vanish due to Lorentz invariance (that is untouched when including powers of $\tilde{q}^2$), while the coefficients d, c, b generate the following extra integrals [11]

$$
\begin{aligned}
\int d^n \bar{q} \frac{\tilde{q}^2}{\bar{D}_i \bar{D}_j} &= -\frac{i\pi^2}{2}\left[m_i^2 + m_j^2 - \frac{(p_i - p_j)^2}{3}\right] + \mathcal{O}(\epsilon)\,, \\
\int d^n \bar{q} \frac{\tilde{q}^2}{\bar{D}_i \bar{D}_j \bar{D}_k} &= -\frac{i\pi^2}{2} + \mathcal{O}(\epsilon)\,, \\
\int d^n \bar{q} \frac{\tilde{q}^4}{\bar{D}_i \bar{D}_j \bar{D}_k \bar{D}_l} &= -\frac{i\pi^2}{6} + \mathcal{O}(\epsilon)\,.
\end{aligned} \tag{4.2}
$$

One can prove that

$$
\begin{aligned}
b(ij;\tilde{q}^2) &= b(ij) + \tilde{q}^2 b^{(2)}(ij)\,, \\
c(ijk;\tilde{q}^2) &= c(ijk) + \tilde{q}^2 c^{(2)}(ijk)\,.
\end{aligned} \tag{4.3}
$$

Furthermore, by using eq. (4.1), the first line of eq. (3.1) becomes

$$
\mathcal{D}^{(m)}(q,\tilde{q}^2) \equiv \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0i_1i_2i_3;\tilde{q}^2) + \tilde{d}(q;i_0i_1i_2i_3;\tilde{q}^2)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} \bar{D}_i\,, \tag{4.4}
$$

and the following expansion holds

$$
\mathcal{D}^{(m)}(q,\tilde{q}^2) = \sum_{j=2}^{m} \tilde{q}^{(2j-4)} d^{(2j-4)}(q)\,, \tag{4.5}
$$

where the last coefficient is independent on q

$$
d^{(2m-4)}(q) = d^{(2m-4)}\,. \tag{4.6}
$$

In practice, once the 4-dimensional coefficients have been determined, one simply redoes the fits for different values of $\tilde{q}^2$, in order to determine $b^{(2)}(ij)$, $c^{(2)}(ijk)$ and $d^{(2m-4)}$. Such three quantities are the coefficients of the three extra scalar integrals listed in eq. (4.2), respectively, so that

$$
\begin{aligned}
\mathrm{R}_1 = &-\frac{i}{96\pi^2} d^{(2m-4)} - \frac{i}{32\pi^2} \sum_{i_0<i_1<i_2}^{m-1} c^{(2)}(i_0i_1i_2) \\
&-\frac{i}{32\pi^2} \sum_{i_0<i_1}^{m-1} b^{(2)}(i_0i_1)\left(m_{i_0}^2 + m_{i_1}^2 - \frac{(p_{i_0} - p_{i_1})^2}{3}\right)\,.
\end{aligned} \tag{4.7}
$$

A formula similar to eq. (4.7) has also been derived in [9].

In appendix A, we prove eqs. (4.3)–(4.6), we single out the origin of $d^{(2m-4)}$ as the coefficient of the last integral of eq. (4.2) and we show how it can be also derived outside the

OPP technique. Finally, yet another way of computing $d^{(2m-4)}$ can be obtained by noticing that

$$d^{(2m-4)} = \lim_{\tilde{q}^2 \to \infty} \frac{\mathcal{D}^{(m)}(q, \tilde{q}^2)}{\tilde{q}^{(2m-4)}} \,. \tag{4.8}$$

This last method is also implemented in the code of ref. [13].

We stress that the way of computing the coefficients appearing in eq. (4.7) is immaterial. Therefore the method to extract R_1 described in this section, namely by looking at the mass dependence of the coefficients of the scalar loop functions, can be used independently on the OPP technique. In particular, one can derive all needed coefficients also with the help of analytical methods.

We close this section by making contact between our way to compute the Rational Part of the amplitude and the relevant literature. The procedure of finding the RTs by fitting the various powers of $\tilde{q}^2$ in the coefficients of the cut constructible part has been first introduced in [11], for up to 4-point functions and, for the general case, in [13]. The authors of [9] (GKM), also apply a similar technique, but with two differences, that we will now comment in turn.

- GKM do not split R into 2 parts, as we do. This is performed at the price of introducing n-dimensional polarization vectors to explicitly compute amplitudes in n dimensions. As far as we understand, such a procedure has been shown to work only in the case purely gluon amplitudes, for which the extra polarization can be treated like a 'scalar' particle. How to extend such a technique to fermions is unclear. In our approach, the full procedure is, instead, 4-dimensional and fully general, avoiding this kind of complications. In our case the price to pay is splitting R into 2 pieces (R_1 and R_2) and deriving extra tree-level like Feynman rules to compute R_2.
- GKM introduce a pentuple cut, while we don't. The reason is that, as we prove in appendix A, the $\tilde{q}^2$ dependence of the coefficients d of the 4-point scalar functions, coming from expressing 5-point scalar functions in terms of boxes, cannot generate RTs. However, combining those contributions together with the 4-point coefficients generated by the *pure* 4-point part of the amplitude, all d's become rational functions of $\tilde{q}^2$, instead of polynomial ones. Therefore, in order to avoid uneasy-to-handle rational functions of $\tilde{q}^2$, GKM *subtract* the pentagon contributions by explicitly computing pentuple residues, while we *recombine*, instead, all 4-,5-, $\cdots$, m-point structures together to get back the polynomial dependence of $\tilde{q}^2$ given in eq. (4.5).

5. Conclusions

We have discussed and clarified the origin of the Rational Terms appearing in one-loop amplitudes, showing that they can be classified in two classes. The first class (R_2) can be computed by defining tree-level like Feynman rules for the theory at hand. We precisely outlined the way to derive the needed extra Feynman rules, listing them explicitly in the case of QED. We therefore consider the problem of computing R_2 completely solved. The second piece (R_1) can be calculated in two different ways. We presented a first technique that relies on the OPP method and a second, more general, computational strategy. Both methods have been successfully tested within the OPP method. The second one, however, is more suitable for a numerical implementation, and it has been used in the numerical code CutTools.

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A. The $\tilde{q}^2$ dependence of the OPP coefficients

Our starting point is a rank r tensor m-point *integrand* defined as

$$A_{m;r} \equiv \frac{q_{\mu_1} \cdots q_{\mu_r}}{\bar{D}_0 \cdots \bar{D}_{m-1}} \quad (m > 3\,,\; r > 1)\,. \tag{A.1}$$

By expressing the integration momentum q in terms of the basis of the external vectors, with coefficients linearly depending on the propagator function appearing in the denominator [11], one ends up in an expression that it is identical to the OPP master equation plus terms containing higher-point scalar amplitudes $A_{k;0}$ $k = 5, \ldots, m$. Since all reductions formula used so far are simple polynomial in terms of $\tilde{q}^2$, the same is true for all coefficients appearing in that expansion. This, in conjunction with the fact that the maximum rank allowed $r = m$, easily proves first of all eq. (4.3). In the next step one reduces the $k-$point scalar terms $k = 5, \ldots, m$ in terms of the $4-$point ones at the integrand level. After that step the individual d and $\tilde{d}$ coefficients become rational functions of $\tilde{q}^2$. Nevertheless, since the $\mathcal{D}^{(m)}(q, \tilde{q}^2)$ of eqs. (4.4) and (4.5) is nothing more than the combined numerator of all scalar terms with $k = 4, \ldots, m$, and all coefficients are polynomials in $\tilde{q}^2$, so is the $\mathcal{D}^{(m)}(q, \tilde{q}^2)$ function. Finally it is straightforward to see that in the OPP expansion in terms of $n-$dimensional propagators, the $\tilde{q}^2 \to \infty$ behavior of the individual d terms is $\tilde{q}^4$, which means that the only integral involved is the last one of eq. (4.2).

Another way to derive the same results is by using the reduction at the *integrand* level introduced in [14]. One can express $A_{m;r}$ as a linear combination, with tensor coefficients, of five classes of lower rank tensors: $A_{m;r-1}, \tilde{q}^2 A_{m;r-2}, A_{m;r-2}, A_{m-1;r-1}$ and $A_{m-1;r-2}$. To keep things as transparent as possible, we omit explicitly writing the coefficients and we denote such a linear combination using the following notation[3]

$$A_{m;r} = \left\{A_{m;r-1}|\tilde{q}^2 A_{m;r-2}|A_{m;r-2}|A_{m-1;r-1}|A_{m-1;r-2}\right\} . \tag{A.2}$$

Analogously, it was proven that

$$\begin{aligned} A_{m;1} &= \{A_{m;0}|A_{m-1;0}\} \qquad (m > 4) , \\ A_{4;1} &= \left\{A_{4;0}|A_{3;0}|\tilde{d}(q)A_{4;0}\right\} , \end{aligned} \tag{A.3}$$

where $\tilde{d}(q)$ is defined such as

$$\int d^n \, \bar{q} \, \tilde{d}(q) A_{4;0} = 0 . \tag{A.4}$$

By using eqs. (A.2) and (A.3) it is easy to constructively prove eqs. (4.5) and (4.6). We explicitly give the derivation for the case $m = 6$ and $r = 6$. By iteratively applying eqs. (A.2) and (A.3), one ends up with

$$A_{6;6} = \sum_{j=4,5,6} \left\{A_{j;0}|\tilde{q}^2 A_{j;0}|\tilde{q}^4 A_{j;0}\right\} + \left\{\tilde{q}^6 A_{6;0}\right\} + \left\{\tilde{d}(q)A_{4;0}|\tilde{d}(q)\tilde{q}^2 A_{4;0}\right\} + \mathcal{O}(A_{3;r_3}) , \tag{A.5}$$

where $\mathcal{O}(A_{3;r_3})$ means that we are neglecting contributions with 3 or less denominators. By power counting, only the term $\tilde{q}^4 A_{4;0}$ contributes to R_1. Notice also that its coefficient (that we call z_4) is independent on q. Now we can take a common denominator in eq. (A.5) by multiplying and dividing 5 and 4-point structures by the relevant missing n-dimensional propagators. In particular, for example, by calling $\bar{D}_i$ and $\bar{D}_j$ the 2 denominators that do not appear in $A_{4;0}$

$$z_4 \, \tilde{q}^4 A_{4;0} = z_4 \tilde{q}^4 \bar{D}_i \bar{D}_j A_{6;0} = z_4 \tilde{q}^4 (\tilde{q}^2 + D_i)(\tilde{q}^2 + D_j) A_{6;0} . \tag{A.6}$$

The numerator of the resulting expression is polynomial in $\tilde{q}^2$ and it is nothing but the function $\mathcal{D}^{(m)}(q, \tilde{q}^2)$ of eqs. (4.4) and (4.5), with $m = 6$. Furthermore $d^{(8)} = z_4$, independent on q. The general case can be derived along the same lines.

Eq. (A.6) also clarifies why $d^{(2m-4)}$ is the coefficient of the 4-point like last integrals of eq. (4.2). In fact, $m-4$ among the m original n-dimensional denominators always completely factorize in front of $d^{(2m-4)}$. Notice also that the origin of the coefficient $d^{(2m-4)}$ is uniquely coming from the mass dependence of the coefficients of the 4-point scalar functions after tensor reduction, but *before expressing m-point scalar functions with $m > 4$ in terms of boxes.* The reason why we do not reduce eq. (A.5) to structures with 4-denominators is that this would bring a $\tilde{q}^2$ dependence in the denominator, when passing from 5 to 4 denominators. In our notation [15]

$$A_{5;0} = \left\{ \frac{1}{\tilde{q}^2 + c_i} A_{4;0} \middle| \frac{1}{\tilde{q}^2 + c_i} \tilde{d}(q) A_{4;0} \right\} , \tag{1.7}$$

with c_i constants. It is therefore much better to take, instead, common denominators. Finally, analogous techniques can be used to prove eq. (4.3).

[3] Notice that each of the five terms of eq. (A.2) may actually represent an entire class of contributions with different combinations of denominators. For example $A_{m-1;r-1}$ stands for all m rank $r-1$ tensor *integrands* that can be obtained by omitting 1 among the original m possible denominators.

References

[1] R.K. Ellis, W.T. Giele and G. Zanderighi, *The one-loop amplitude for six-gluon scattering*, *JHEP* **05** (2006) 027 [hep-ph/0602185];
R. Britto, B. Feng and P. Mastrolia, *The cut-constructible part of QCD amplitudes*, *Phys. Rev.* **D 73** (2006) 105004 [hep-ph/0602178];
C.F. Berger, Z. Bern, L.J. Dixon, D. Forde and D.A. Kosower, *Bootstrapping one-loop QCD amplitudes with general helicities*, *Phys. Rev.* **D 74** (2006) 036009 [hep-ph/0604195];
Z. Bern, N.E.J. Bjerrum-Bohr, D.C. Dunbar and H. Ita, *Recursive calculation of one-loop QCD integral coefficients*, *JHEP* **11** (2005) 027 [hep-ph/0507019];
T. Binoth, J.P. Guillet, G. Heinrich, E. Pilon and C. Schubert, *An algebraic/numerical formalism for one-loop multi-leg amplitudes*, *JHEP* **10** (2005) 015 [hep-ph/0504267];
J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *Non-supersymmetric loop amplitudes and MHV vertices*, *Nucl. Phys.* **B 712** (2005) 59 [hep-th/0412108];
G. Bélanger et al., *Full $O(\alpha)$ electroweak corrections to double Higgs- strahlung at the linear collider*, *Phys. Lett.* **B 576** (2003) 152 [hep-ph/0309010];
A. Denner, S. Dittmaier, M. Roth and M.M. Weber, *Radiative corrections to Higgs-boson production in association with top-quark pairs at e^+e^- colliders*, *Nucl. Phys.* **B 680** (2004) 85 [hep-ph/0309274];
A. Denner, S. Dittmaier, M. Roth and L.H. Wieders, *Electroweak corrections to charged-current $e^+e^- \to 4$ fermion processes: technical details and further results*, *Nucl. Phys.* **B 724** (2005) 247 [hep-ph/0505042]; *Complete electroweak $O(\alpha)$ corrections to charged- current $e^+e^- \to 4$ fermion processes*, *Phys. Lett.* **B 612** (2005) 223 [hep-ph/0502063];
K. Kato et al., *Radiative corrections for Higgs study at the ILC*, PoS(HEP2005)312;
T. Binoth, G. Heinrich, T. Gehrmann and P. Mastrolia, *Six-photon amplitudes*, *Phys. Lett.* **B 649** (2007) 422 [hep-ph/0703311];
S. Weinzierl, *Automated calculations for multi-leg processes*, arXiv:0707.3342;
D. Maître and P. Mastrolia, *S@M, a Mathematica implementation of the Spinor-Helicity formalism*, arXiv:0710.5559;
Z. Nagy and D.E. Soper, *Numerical integration of one-loop Feynman diagrams for N- photon amplitudes*, *Phys. Rev.* **D 74** (2006) 093006 [hep-ph/0610028];
A. Ferroglia, M. Passera, G. Passarino and S. Uccirati, *All-purpose numerical evaluation of one-loop multi-leg Feynman diagrams*, *Nucl. Phys.* **B 650** (2003) 162 [hep-ph/0209219];
R. Pittau, *A simple method for multi-leg loop calculations*, *Comput. Phys. Commun.* **104** (1997) 23 [hep-ph/9607309]; *A simple method for multi-leg loop calculations. II: a general algorithm*, *Comput. Phys. Commun.* **111** (1998) 48 [hep-ph/9712418].

[2] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, *Nucl. Phys.* **B 435** (1995) 59 [hep-ph/9409265]; *One loop n point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B 425** (1994) 217;
Z. Bern, L.J. Dixon and D.A. Kosower, *On-shell methods in perturbative QCD*, *Ann. Phys. (NY)* **322** (2007) 1587 [arXiv:0704.2798].

[3] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $N = 4$ super-Yang-Mills*, *Nucl. Phys.* **B 725** (2005) 275 [hep-th/0412103].

[4] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [hep-th/0312171];
F. Cachazo, P. Svrček and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, *JHEP* **09** (2004) 006 [hep-th/0403047];

A. Brandhuber, B.J. Spence and G. Travaglini, *One-loop gauge theory amplitudes in $N = 4$ super Yang-Mills from MHV vertices*, *Nucl. Phys.* **B 706** (2005) 150 [hep-th/0407214];
F. Cachazo, P. Svrček and E. Witten, *Twistor space structure of one-loop amplitudes in gauge theory*, *JHEP* **10** (2004) 074 [hep-th/0406177];
I. Bena, Z. Bern, D.A. Kosower and R. Roiban, *Loops in twistor space*, *Phys. Rev.* **D 71** (2005) 106010 [hep-th/0410054].

[5] R.K. Ellis, W.T. Giele and Z. Kunszt, *A numerical unitarity formalism for evaluating one-loop amplitudes*, *JHEP* **03** (2008) 003 [arXiv:0708.2398];
C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, *Unitarity cuts and reduction to master integrals in d dimensions for one-loop amplitudes*, *JHEP* **03** (2007) 111 [hep-ph/0612277]; *D-dimensional unitarity cut method*, *Phys. Lett.* **B 645** (2007) 213 [hep-ph/0609191];
D. Forde, *Direct extraction of one-loop integral coefficients*, *Phys. Rev.* **D 75** (2007) 125019 [arXiv:0704.1835];
W.B. Kilgore, *One-loop integral coefficients from generalized unitarity*, arXiv:0711.5015;
N.E.J. Bjerrum-Bohr, D.C. Dunbar and W.B. Perkins, *Analytic structure of three-mass triangle coefficients*, *JHEP* **04** (2008) 038 [arXiv:0709.2086].

[6] T. Binoth, J.P. Guillet and G. Heinrich, *Algebraic evaluation of rational polynomials in one-loop amplitudes*, *JHEP* **02** (2007) 013 [hep-ph/0609054].

[7] Z. Xiao, G. Yang and C.-J. Zhu, *The rational part of QCD amplitude. I: the general formalism*, *Nucl. Phys.* **B 758** (2006) 1 [hep-ph/0607015];
X. Su, Z. Xiao, G. Yang and C.-J. Zhu, *The rational part of QCD amplitude. II: the five-gluon*, *Nucl. Phys.* **B 758** (2006) 35 [hep-ph/0607016].

[8] Z. Bern, L.J. Dixon and D.A. Kosower, *Bootstrapping multi-parton loop amplitudes in QCD*, *Phys. Rev.* **D 73** (2006) 065013 [hep-ph/0507005];
S.D. Badger, E.W.N. Glover and K. Risager, *One-loop phi-MHV amplitudes using the unitarity bootstrap*, *JHEP* **07** (2007) 066 [arXiv:0704.3914].

[9] W.T. Giele, Z. Kunszt and K. Melnikov, *Full one-loop amplitudes from tree amplitudes*, *JHEP* **04** (2008) 049 [arXiv:0801.2237].

[10] R. Britto and B. Feng, *Unitarity cuts with massive propagators and algebraic expressions for coefficients*, *Phys. Rev.* **D 75** (2007) 105006 [hep-ph/0612089]; *Integral coefficients for one-loop amplitudes*, *JHEP* **02** (2008) 095 [arXiv:0711.4284].

[11] G. Ossola, C.G. Papadopoulos and R. Pittau, *Reducing full one-loop amplitudes to scalar integrals at the integrand level*, *Nucl. Phys.* **B 763** (2007) 147 [hep-ph/0609007].

[12] G. Ossola, C.G. Papadopoulos and R. Pittau, *Numerical evaluation of six-photon amplitudes*, *JHEP* **07** (2007) 085 [arXiv:0704.1271].

[13] G. Ossola, C.G. Papadopoulos and R. Pittau, *CutTools: a program implementing the OPP reduction method to compute one-loop amplitudes*, *JHEP* **03** (2008) 042 [arXiv:0711.3596].

[14] F. del Aguila and R. Pittau, *Recursive numerical calculus of one-loop tensor integrals*, *JHEP* **07** (2004) 017 [hep-ph/0404120];
R. Pittau, *Formulae for a numerical computation of one-loop tensor integrals*, hep-ph/0406105.

[15] W.L. van Neerven and J.A.M. Vermaseren, *Large loop integrals*, *Phys. Lett.* **B 137** (1984) 241;
A. Denner and S. Dittmaier, *Reduction of one-loop tensor 5-point integrals*, *Nucl. Phys.* **B 658** (2003) 175 [hep-ph/0212259]; *Reduction schemes for one-loop tensor integrals*, *Nucl. Phys.* **B 734** (2006) 62 [hep-ph/0509141].